

The Kastler-Kalau-Walze type theorem and the spectral action for perturbations of Dirac operators

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Abstract

In this paper, we prove the Kastler-Kalau-Walze type theorem for perturbations of Dirac operators for manifolds with or without boundary. As a corollary, we give two kinds of operator-theoretic explanations of the gravitational action on the boundary. We also compute the spectral action for Dirac operators with two form perturbations on 4-dimensional manifolds.

Keywords: Perturbations of Dirac operators; noncommutative residue; gravitational action; spectral action; Seeley-dewitt coefficients

1 Introduction

The noncommutative residue found in [Gu] and [Wo] plays a prominent role in noncommutative geometry. In [Co1], Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. In [Co2], Connes proved that the noncommutative residue on a compact manifold M coincided with the Dixmier's trace on pseudodifferential operators of order $-\dim M$. Several years ago, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action, which was called Kastler-Kalau-Walze Theorem now. In [Ka], Kastler gave a brute-force proof of this theorem. In [KW], Kalau and Walze proved this theorem by the normal coordinates way simultaneously. In [Ac], Ackermann gave a note on a new proof of this theorem by the heat kernel expansion way. The Kastler-Kalau-Walze theorem had been generalized to some cases, for example, Dirac operators with torsion [AT], CR manifolds [Po], \mathbb{R}^n [BC].

On the other hand, Fedosov etc. defined a noncommutative residue on Boutet de Monvel's algebra and proved that it was a unique continuous trace in [FGLS]. In [Sc], Schrohe gave the relation between the Dixmier trace and the noncommutative residue for manifolds with boundary. In [Wa1],[Wa2], we give an operator theoretic explanation of the gravitational action for manifolds with boundary and prove a Kastler-Kalau-Walze type theorem for Dirac operators and signature operators on manifolds with boundary.

Connes's spectral action principle ([Co3]) in noncommutative geometry states that the physical action depends only on the spectrum. We assume that space-time is a product of a continuous manifold and a finite space. The spectral action is defined as the trace of an arbitrary function of the Dirac operator for the bosonic part and a Dirac type action of the fermionic part including all their interactions. In [CC1], Chamseddine and Connes computed the Spectral action for Dirac operators on spin manifolds and the Chamseddine-Connes spectral action comprises the Einstein-Hilbert action of general relativity and the bosonic part of the action of the standard model of particle physics. In [HPS], Hanisch, Pfäffle and Stephan derived a formula for the gravitational part of the spectral action for Dirac operators on 4-dimensional spin manifolds with totally anti-symmetric torsion. They also deduced the Lagrangian for the Standard Model of particle Physics in the presence of torsion from the Chamseddine-Connes spectral action. In [CC2], Chamseddine and Connes studied the spectral action for spin manifolds with boundary and generalized this action to noncommutative spaces which are products of a spin manifold and a finite space. In [IL], Iochum and Levy studied the spectral action for Dirac operators with one form perturbations and proved that there were no tadpoles for compact spin manifolds without boundary. In [SZ], they investigated the spectral action for scalar perturbations of Dirac operators. Motivated by [IL], [SZ] and [HPS], we studied the Dirac operators with general perturbations. We prove the Kastler-Kalau-Walze type theorem for general perturbations of Dirac operators for manifolds with or without boundary. By perturbations, we can give two kinds of operator-theoretic explanations of the gravitational action on the boundary. We also compute the spectral action for Dirac operators with two form perturbations on 4-dimensional manifolds and give detailed computations of spectral action for scalar perturbations of Dirac operators in [SZ].

This paper is organized as follows: In Section 2, we prove the Lichnerowicz formula for perturbations of Dirac operators and prove the Kastler-Kalau-Walze type theorem for perturbations of Dirac operators for manifolds with or without boundary. In Section 3, we prove the Kastler-Kalau-Walze type theorem for conformal perturbations of Dirac operators for manifolds with or without boundary. In Section 4, We compute the spectral action for Dirac operators with scalar and two form perturbations on 4-dimensional manifolds.

2 The Kastler-Kalau-Walze type theorem for perturbations of Dirac operators

2.1 The Kastler-Kalau-Walze type theorem for perturbations of Dirac operators for manifolds without boundary

Let M be a smooth compact Riemannian n -dimensional manifold without boundary and V be a vector bundle on M . Recall that a differential operator P is of Laplace

type if it has locally the form

$$P = -(g^{ij}\partial_i\partial_j + A^i\partial_i + B), \quad (2.1)$$

where $(g^{ij})_{1 \leq i, j \leq n}$ is the inverse matrix associated to the metric g on M , and A^i and B are smooth sections of $\text{End}(V)$ on M (endomorphism). If P is a Laplace type operator of the form (2.1), then (see [Gi]) there is an unique connection ∇ on V and an unique endomorphism E such that

$$P = -[g^{ij}(\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\nabla_{\partial_i}^L\partial_j}) + E], \quad (2.2)$$

where ∇^L denotes the Levi-civita connection on M . Moreover (with local frames of T^*M and V), $\nabla = dx^i \otimes (\partial_i + \omega_i)$ and E are related to g^{ij} , A^i and B through

$$\omega_i = \frac{1}{2}g_{ij}(A^j + g^{kl}\Gamma_{kl}^j\text{Id}), \quad (2.3)$$

$$E = B - g^{ij}(\partial_i(\omega_j) + \omega_i\omega_j - \omega_k\Gamma_{ij}^k), \quad (2.4)$$

where Γ_{ij}^k are the Christoffel coefficients of ∇^L .

Now we let M be a n -dimensional oriented spin manifold with Riemannian metric g . We recall that the Dirac operator D is locally given as follows in terms of orthonormal frames e_i , $1 \leq i \leq n$ and natural frames ∂_i of TM : one has

$$D = \sum_{i,j} g^{ij}c(\partial_i)\nabla_{\partial_j}^S = \sum_i c(e_i)\nabla_{e_i}^S, \quad (2.5)$$

where

$$\nabla_{\partial_i}^S = \partial_i + \sigma_i, \quad \sigma_i = \frac{1}{4} \sum_{jk} \langle \nabla_{\partial_i}^L e_j, e_k \rangle c(e_j)c(e_k). \quad (2.6)$$

Let

$$\partial^j = g^{ij}\partial_i, \quad \sigma^i = g^{ij}\sigma_j, \quad \Gamma^k = g^{ij}\Gamma_{ij}^k. \quad (2.7)$$

By (6a) in [Ka], we have

$$D^2 = -g^{ij}\partial_i\partial_j - 2\sigma^j\partial_j + \Gamma^k\partial_k - g^{ij}[\partial_i(\sigma_j) + \sigma_i\sigma_j - \Gamma_{ij}^k\sigma_k] + \frac{1}{4}s, \quad (2.8)$$

where s is the scalar curvature. Let Ψ be a smooth differential form on M and we also denote the associated Clifford action by Ψ . We will compute $D_\Psi^2 := (D + \Psi)^2$. we note that

$$(D + \Psi)^2 = D^2 + D\Psi + \Psi D + \Psi^2, \quad (2.9)$$

$$D\Psi + \Psi D = \sum_{ij} g^{ij} (c(\partial_i)\Psi + \Psi c(\partial_i)) \partial_j + \sum_{ij} g^{ij} (c(\partial_i)\partial_j(\Psi) + c(\partial_i)\sigma_j\Psi + \Psi c(\partial_i)\sigma_j). \quad (2.10)$$

By (2.8)-(2.10), we have

$$D_\Psi^2 = -g^{ij}\partial_i\partial_j + (-2\sigma^j + \Gamma^j + c(\partial^j)\Psi + \Psi c(\partial^j)) \partial_j$$

$$+g^{ij}[-\partial_i(\sigma_j) - \sigma_i\sigma_j + \Gamma_{ij}^k\sigma_k + c(\partial_i)\partial_j(\Psi) + c(\partial_i)\sigma_j\Psi + \Psi c(\partial_i)\sigma_j] + \frac{1}{4}s + \Psi^2. \quad (2.11)$$

By (2.11), (2.3) and (2.4), we have

$$\omega_i = \sigma_i - \frac{1}{2}[c(\partial_i)\Psi + \Psi c(\partial_i)], \quad (2.12)$$

$$\begin{aligned} E = & -c(\partial_i)\partial^i(\Psi) - c(\partial_i)\sigma^i\Psi - \Psi c(\partial_i)\sigma^i - \frac{1}{4}s - \Psi^2 + \frac{1}{2}\partial^j[c(\partial_j)\Psi + \Psi c(\partial_j)] \\ & - \frac{1}{2}\Gamma^k[c(\partial_k)\Psi + \Psi c(\partial_k)] + \frac{1}{2}\sigma^j[c(\partial_j)\Psi + \Psi c(\partial_j)] \\ & + \frac{1}{2}[c(\partial_j)\Psi + \Psi c(\partial_j)]\sigma^j - \frac{g^{ij}}{4}[c(\partial_i)\Psi + \Psi c(\partial_i)][c(\partial_j)\Psi + \Psi c(\partial_j)]. \end{aligned} \quad (2.13)$$

So

$$\nabla_X = \nabla_X^S - \frac{1}{2}[c(X)\Psi + \Psi c(X)]. \quad (2.14)$$

Since E is globally defined on M , so we can compute E in normal coordinates. Taking normal coordinates about x_0 , then $\sigma^i(x_0) = 0$, $\partial^j[c(\partial_j)](x_0) = 0$, $\Gamma^k(x_0) = 0$, $g^{ij}(x_0) = \delta_i^j$. So

$$\begin{aligned} E(x_0) = & -\frac{1}{4}s - \Psi^2 + \frac{1}{2}[\partial^j(\Psi)c(\partial_j) - c(\partial_j)\partial^j(\Psi)] - \frac{1}{4}[c(\partial_i)\Psi + \Psi c(\partial_i)]^2(x_0) \\ = & -\frac{1}{4}s - \Psi^2 + \frac{1}{2}[e_j(\Psi)c(e_j) - c(e_j)e_j(\Psi)] - \frac{1}{4}[c(e_i)\Psi + \Psi c(e_i)]^2(x_0) \\ = & -\frac{1}{4}s - \Psi^2 + \frac{1}{2}[\nabla_{e_j}^S(\Psi)c(e_j) - c(e_j)\nabla_{e_j}^S(\Psi)] - \frac{1}{4}[c(e_i)\Psi + \Psi c(e_i)]^2(x_0). \end{aligned} \quad (2.15)$$

And we get the following Lichnerowicz formula:

Theorem 2.1

$$D_\Psi^2 = -[g^{ij}(\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\nabla_{\partial_i}^L\partial_j})] + \frac{1}{4}s + \Psi^2 - \frac{1}{2}[\nabla_{e_j}^S(\Psi)c(e_j) - c(e_j)\nabla_{e_j}^S(\Psi)] + \frac{1}{4}[c(e_i)\Psi + \Psi c(e_i)]^2, \quad (2.16)$$

where ∇_{∂_i} is defined by (2.14).

We see two special cases of Theorem 2.1. When $\Psi = f$, we have

$$\nabla_X = \nabla_X^S - fc(X); \quad E = -\frac{1}{4}s + (n-1)f^2. \quad (2.17)$$

Corollary 2.2

$$D_f^2 = -[g^{ij}(\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\nabla_{\partial_i}^L\partial_j})] + \frac{1}{4}s + (1-n)f^2. \quad (2.18)$$

When $\Psi = \sqrt{-1}c(X)$ and $X = a_i e^i$ where a_i is a smooth real function and e^i is a dual orthonormal frame by parallel transport along geodesic, then $\nabla_Y = \nabla_Y^S + \sqrt{-1}g(X, Y)$, and by $e_j(c(e_i)) = 0$ and $de^l(x_0) = 0$ (see [BGV, Lemma 4.13]),

$$\begin{aligned} E(x_0) &= -\frac{1}{4}s - |X|^2 + \frac{\sqrt{-1}}{2}[e_j(a^k)c(e_k)c(e_j) - c(e_j)c(e_k)e_j(a^k)] + \frac{1}{4}[c(e_i)c(X) + c(X)c(e_i)]^2 \\ &= -\frac{1}{4}s + \frac{\sqrt{-1}}{2}e_j(a^k)[c(e_k)c(e_j) - c(e_j)c(e_k)] \\ &= -\frac{1}{4}s + \sqrt{-1}\sum_{k \neq j} e_j(a^k)c(e_k)c(e_j)(x_0) = -\frac{1}{4}s - \sqrt{-1}c(dX)(x_0). \end{aligned} \quad (2.19)$$

Corollary 2.3

$$(D + \sqrt{-1}c(X))^2 = -[g^{ij}(\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\nabla_{\partial_i}^L\partial_j})] + \frac{1}{4}s + \sqrt{-1}c(dX). \quad (2.20)$$

When Ψ is a two form, we let $\Psi = 2\sum_{k < l} a_{kl}e^k \wedge e^l = \sum a_{kl}e^k \wedge e^l$, where $a_{kl} = -a_{lk}$, and $c(\Psi) = \sum a_{kl}c(e_k)c(e_l)$. So

$$\nabla_{e_i} = e_i + \frac{1}{4}\sum_{s,t} \omega_{st}(e_i)c(e_s)c(e_t) - \sum_{k,l \neq i} a_{kl}c(e_k)c(e_l)c(e_i), \quad (2.21)$$

where $\omega_{st}(e_i)$ denotes the connection coefficient. By (2.15),

$$\begin{aligned} E &= -\frac{1}{4}s - [a_{kl}c(e_k)c(e_l)]^2 + \frac{1}{2}\{e_j(a_{kl})[c(e_k)c(e_l)c(e_j) - c(e_j)c(e_k)c(e_l)]\} \\ &\quad - \frac{1}{4}[a_{kl}(c(e_i)c(e_k)c(e_l) + c(e_k)c(e_l)c(e_i))]^2, \end{aligned} \quad (2.22)$$

Let $\dim(S(TM)) = d$, by for $k \neq l, \tilde{k} \neq \tilde{l}$

$$\text{Tr}[c(e_k)c(e_l)c(e_{\tilde{k}})c(e_{\tilde{l}})] = d(-\tilde{\delta}_{\tilde{k}}^{\tilde{k}}\tilde{\delta}_{\tilde{l}}^{\tilde{l}} + \tilde{\delta}_{\tilde{k}}^{\tilde{l}}\tilde{\delta}_{\tilde{l}}^{\tilde{k}}), \quad (2.23)$$

we have

$$\text{Tr}\{[a_{kl}c(e_k)c(e_l)]^2\} = -2da_{kl}^2. \quad (2.24)$$

Since the trace of the product of odd Clifford elements is zero, we have

$$\text{Tr}\left[\frac{1}{2}\{e_j(a_{kl})[c(e_k)c(e_l)c(e_j) - c(e_j)c(e_k)c(e_l)]\}\right] = 0. \quad (2.25)$$

$$\begin{aligned} &\text{Tr}\left\{[a_{kl}(c(e_i)c(e_k)c(e_l) + c(e_k)c(e_l)c(e_i))]^2\right\} \\ &= a_{kl}a_{\tilde{k}\tilde{l}}\text{Tr}[c(e_k)c(e_l)c(e_{\tilde{k}})c(e_{\tilde{l}})c(e_i)^2 + c(e_k)c(e_l)c(e_i)^2c(e_{\tilde{k}})c(e_{\tilde{l}}) \\ &\quad + c(e_i)c(e_k)c(e_l)c(e_i)c(e_{\tilde{k}})c(e_{\tilde{l}}) + c(e_k)c(e_l)c(e_i)c(e_{\tilde{k}})c(e_{\tilde{l}})c(e_i)] \\ &= -2nda_{kl}a_{\tilde{k}\tilde{l}}(-\tilde{\delta}_{\tilde{k}}^{\tilde{k}}\tilde{\delta}_{\tilde{l}}^{\tilde{l}} + \tilde{\delta}_{\tilde{k}}^{\tilde{l}}\tilde{\delta}_{\tilde{l}}^{\tilde{k}}) - 2\sum_{i \neq k,l} a_{kl}a_{\tilde{k}\tilde{l}}\text{Tr}[c(e_k)c(e_l)c(e_{\tilde{k}})c(e_{\tilde{l}})] \end{aligned}$$

$$\begin{aligned}
& +2 \sum_{i=k} a_{kl} a_{\widetilde{kl}} \text{Tr} [c(e_k) c(e_l) c(e_{\widetilde{k}}) c(e_{\widetilde{l}})] + 2 \sum_{i=l} a_{kl} a_{\widetilde{kl}} \text{Tr} [c(e_k) c(e_l) c(e_{\widetilde{k}}) c(e_{\widetilde{l}})] \\
& = 4n d a_{kl}^2 + (n-2) 4 d a_{kl}^2 - 4 d a_{kl}^2 - 4 d a_{kl}^2 = 8(n-2) d a_{kl}^2.
\end{aligned} \tag{2.26}$$

By (2.22) and (2.24)-(2.26), we have

$$\text{tr} E = -\frac{1}{4}s + (6-2n)|\Psi|^2 \tag{2.27}$$

and

Corollary 2.4 *let $\Psi = \sum a_{kl} e^k \wedge e^l$ and $a_{kl} = -a_{lk}$, then $\text{tr} E = -\frac{1}{4}s + (6-2n)|\Psi|^2$.*

By (2.15) and $\text{Tr}(AB) = \text{Tr}(BA)$, for the general Ψ , we have

$$\begin{aligned}
\text{Tr}(E) &= \text{Tr} \left[-\frac{1}{4}s - \Psi^2 - \frac{1}{4}[c(e_i)\Psi + \Psi c(e_i)]^2 \right] \\
&= \text{Tr} \left[-\frac{1}{4}s - \frac{1}{2}\Psi c(e_i)\Psi c(e_i) + \left(\frac{n}{2} - 1\right)\Psi^2 \right].
\end{aligned} \tag{2.28}$$

By the Kastler-Kalau-Walze theorem (see [Ka],[KW]), we have

$$\text{Wres}(D_{\Psi}^{-n+2}) = \frac{(2\pi)^{\frac{n}{2}}}{(\frac{n}{2}-2)!} \int_M \text{Tr} \left(\frac{1}{6}s + E \right) d\text{vol}_M, \tag{2.29}$$

where Wres denotes the noncommutative residue (see [Wo]). By (2.28) and (2.29), we have

Theorem 2.5 *For even n -dimensional spin manifolds without boundary, the following equality holds*

$$\text{Wres}(D_{\Psi}^{-n+2}) = \frac{(2\pi)^{\frac{n}{2}}}{(\frac{n}{2}-2)!} \int_M \text{Tr} \left[-\frac{1}{12}s - \frac{1}{2}\Psi c(e_i)\Psi c(e_i) + \left(\frac{n}{2} - 1\right)\Psi^2 \right] d\text{vol}_M. \tag{2.30}$$

By Corollary 2.2, we have

Corollary 2.6 *For even n -dimensional spin manifolds without boundary, the following equality holds*

$$\text{Wres}(D_f^{-n+2}) = \frac{(2\pi)^{\frac{n}{2}}d}{(\frac{n}{2}-2)!} \int_M \left[-\frac{1}{12}s + (n-1)f^2 \right] d\text{vol}_M. \tag{2.31}$$

By Corollary 2.3, we have

Corollary 2.7 *For even n -dimensional spin manifolds without boundary and a one-form Ψ , the following equality holds*

$$\text{Wres}(D_{\Psi}^{-n+2}) = -\frac{(2\pi)^{\frac{n}{2}}d}{12 \times (\frac{n}{2} - 2)!} \int_M \text{sdvol}_M. \quad (2.32)$$

By Corollary 2.4, we have

Corollary 2.8 *For even n -dimensional spin manifolds without boundary and a two-form Ψ , the following equality holds*

$$\text{Wres}(D_{\Psi}^{-n+2}) = \frac{(2\pi)^{\frac{n}{2}}}{(\frac{n}{2} - 2)!} \int_M \text{Tr} \left[-\frac{1}{12} s + (6 - 2n)|\Psi|^2 \right] \text{dvol}_M. \quad (2.33)$$

2.2 The Kastler-Kalau-Walze type theorem for perturbations of Dirac operators for manifolds with boundary

Let M be a 4-dimensional compact oriented spin manifold with boundary ∂M and we assume that the metric g^M on M has the following form near the boundary,

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2, \quad (2.34)$$

where $g^{\partial M}$ is the metric on ∂M . $h(x_n) \in C^\infty([0, 1)) = \{\tilde{h}|_{[0, 1)} | \tilde{h} \in C^\infty((-\varepsilon, 1))\}$ for some $\varepsilon > 0$ and satisfies $h(x_n) > 0$, $h(0) = 1$ where x_n denotes the normal directional coordinate. We want to compute $\widetilde{\text{Wres}}[(\pi^+ D_{\Psi}^{-1})^2]$ (for the related definitions, see [Wa1]). By (2.2.4) in [Wa1], we have

$$\widetilde{\text{Wres}}[(\pi^+ D_{\Psi}^{-1})^2] = \int_M \int_{|\xi|=1} \text{trace}_{S(TM)} [\sigma_{-4}(D_{\Psi}^{-2})] \sigma(\xi) dx + \int_{\partial M} \Phi, \quad (2.35)$$

where

$$\begin{aligned} \Phi = & \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \\ & \times \text{trace}_{S(TM)} [\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+(D_{\Psi}^{-1})(x', 0, \xi', \xi_n) \times \partial_{x'}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l(D_{\Psi}^{-1})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx', \end{aligned} \quad (2.36)$$

where the sum is taken over $r - k - |\alpha| + l - j - 1 = -4$, $r, l \leq -1$ and $\sigma_r^+(D_{\Psi}^{-1}) = \pi_{\xi_n}^+ \sigma_r(D_{\Psi}^{-1})$ (for the definition of π^+ , see [Wa1]). By Theorem 2.5, we have

$$\int_M \int_{|\xi|=1} \text{tr} [\sigma_{-4}(D_{\Psi}^{-2})] \sigma(\xi) dx = 4\pi^2 \int_M \text{Tr} \left[-\frac{1}{12} s - \frac{1}{2} \Psi c(e_i) \Psi c(e_i) + \left(\frac{n}{2} - 1\right) \Psi^2 \right] \text{dvol}_M. \quad (2.37)$$

So we only to compute $\int_{\partial M} \Phi$. Similar to Lemma 2.1 in [Wa1], we have

Lemma 2.9

$$q_{-1}(D_{\Psi}^{-1}) = \frac{\sqrt{-1}c(\xi)}{|\xi|^2}; \quad q_{-2}(D_{\Psi}^{-1}) = q_{-2}(D^{-1}) + \frac{c(\xi)\Psi c(\xi)}{|\xi|^4}. \quad (2.38)$$

Similar to the computations in section 2.2.2 in [Wa1], we can compute Φ in five cases. Since $q_{-1}(D_{\Psi}^{-1}) = q_{-1}(D^{-1})$, then cases a (I), (II),(III) in our case are the same with the cases a (I), (II),(III) in [Wa1]. So we only need to compute the case (b) and the case (c). By Lemma 2.9,

$$\begin{aligned} \text{case(b)} &:= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ q_{-2}(D^{-1}) \times \partial_{\xi_n} q_{-1}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &- i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \left(\frac{c(\xi)\Psi c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n} q_{-1}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \quad (2.39)$$

By the case (b) in [Wa1], we have

$$-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ q_{-2}(D^{-1}) \times \partial_{\xi_n} q_{-1}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' = \frac{9}{8} \pi h'(0) \Omega_3 dx', \quad (2.40)$$

where Ω_3 is the canonical volume of 3-dimensional unit sphere.

$$\begin{aligned} \pi_{\xi_n}^+ \left(\frac{c(\xi)\Psi c(\xi)}{|\xi|^4} \right) (x_0)|_{|\xi'|=1} &= \pi_{\xi_n}^+ \left[\frac{[c(\xi') + \xi_n c(dx_n)] \Psi [c(\xi') + \xi_n c(dx_n)]}{(1 + \xi_n^2)^2} \right] \\ &= \frac{1}{2\pi i} \int_{\Gamma^+} \frac{\frac{c(\xi')\Psi c(\xi') + c(dx_n)\Psi c(\xi')\eta_n + c(\xi')\Psi c(dx_n)\eta_n + c(dx_n)\Psi c(dx_n)\eta_n^2}{(\eta_n + i)^2(\xi_n - \eta_n)}}{(\eta_n - i)^2} d\eta_n \\ &= \left[\frac{c(\xi')\Psi c(\xi') + c(dx_n)\Psi c(\xi')\eta_n + c(\xi')\Psi c(dx_n)\eta_n + c(dx_n)\Psi c(dx_n)\eta_n^2}{(\eta_n + i)^2(\xi_n - \eta_n)} \right]^{(1)} \Big|_{\eta_n=i} \\ &= -\frac{i\xi_n + 2}{4(\xi_n - i)^2} c(\xi')\Psi c(\xi') - \frac{i}{4(\xi_n - i)^2} [c(dx_n)\Psi c(\xi') \\ &\quad + c(\xi')\Psi c(dx_n)] - \frac{i\xi_n}{4(\xi_n - i)^2} c(dx_n)\Psi c(dx_n). \end{aligned} \quad (2.41)$$

$$\partial_{\xi_n} q_{-1}|_{|\xi'|=1} = \sqrt{-1} \left[\frac{1 - \xi_n^2}{(1 + \xi_n^2)^2} c(dx_n) - \frac{2\xi_n}{(1 + \xi_n^2)^2} c(\xi') \right]. \quad (2.42)$$

By (2.41) and (2.42) and

$$c(\xi')^2|_{|\xi'|=1} = -1, \quad c(dx_n)^2 = -1, \quad c(\xi')c(dx_n) = -c(dx_n)c(\xi'), \quad \text{Tr}(AB) = \text{Tr}(BA), \quad (2.43)$$

we get

$$\text{trace} \left[\pi_{\xi_n}^+ \left(\frac{c(\xi)\Psi c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n} q_{-1}(D^{-1}) \right] (x_0)|_{|\xi'|=1}$$

$$= \frac{\sqrt{-1}}{2(1+\xi_n^2)^2} \text{Tr}[c(dx_n)\Psi] + \frac{1}{2(1+\xi_n^2)^2} \text{Tr}[c(\xi')\Psi]. \quad (2.43)$$

Considering for $i < n$, $\int_{|\xi'|=1} \xi_i \sigma(\xi') = 0$, then

$$\begin{aligned} -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \left(\frac{c(\xi)\Psi c(\xi)}{|\xi|^4} \right) \times \partial_{\xi_n} q_{-1}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ = \frac{\pi}{4} \Omega_3 \text{Tr}[c(dx_n)\Psi] dx', \end{aligned} \quad (2.44)$$

and

$$\text{case (b)} = \frac{9}{8} \pi h'(0) \Omega_3 dx' + \frac{\pi}{4} \Omega_3 \text{Tr}[c(dx_n)\Psi] dx'. \quad (2.45)$$

$$\begin{aligned} \text{case (c)} &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ q_{-1}(D^{-1}) \times \partial_{\xi_n} q_{-2}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' \\ &-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[\pi_{\xi_n}^+ q_{-1}(D^{-1}) \times \partial_{\xi_n} \left(\frac{c(\xi)\Psi c(\xi)}{|\xi|^4} \right) \right] (x_0) d\xi_n \sigma(\xi') dx'. \end{aligned} \quad (2.46)$$

By the case (c) in [Wa1], we get

$$-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ q_{-1}(D^{-1}) \times \partial_{\xi_n} q_{-2}(D^{-1})](x_0) d\xi_n \sigma(\xi') dx' = -\frac{9}{8} \pi h'(0) \Omega_3 dx'. \quad (2.47)$$

Direct computations show that

$$\pi_{\xi_n}^+ q_{-1}(D^{-1})(x_0)|_{|\xi'|=1} = \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)}, \quad (2.48)$$

$$\begin{aligned} \partial_{\xi_n} \left(\frac{c(\xi)\Psi c(\xi)}{|\xi|^4} \right) = \\ \frac{c(dx_n)\Psi c(\xi') + c(\xi')\Psi c(dx_n) + 2\xi_n c(dx_n)\Psi c(dx_n)}{(1+\xi_n^2)^2} - \frac{4\xi_n c(\xi)\Psi c(\xi)}{(1+\xi_n^2)^3}, \end{aligned} \quad (2.49)$$

and

$$\begin{aligned} \text{trace} \left[\pi_{\xi_n}^+ q_{-1}(D^{-1}) \times \partial_{\xi_n} \left(\frac{c(\xi)\Psi c(\xi)}{|\xi|^4} \right) \right] (x_0)|_{|\xi'|=1} \\ = \frac{-1}{(\xi_n - i)(\xi_n + i)^3} \text{Tr}[c(\xi')\Psi] + \frac{i}{(\xi_n - i)(\xi_n + i)^3} \text{Tr}[c(dx_n)\Psi]. \end{aligned} \quad (2.50)$$

$$\begin{aligned} -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[\pi_{\xi_n}^+ q_{-1}(D^{-1}) \times \partial_{\xi_n} \left(\frac{c(\xi)\Psi c(\xi)}{|\xi|^4} \right) \right] (x_0) d\xi_n \sigma(\xi') dx' \\ = -\frac{\pi}{4} \Omega_3 \text{Tr}[c(dx_n)\Psi] dx', \end{aligned} \quad (2.51)$$

$$\text{case (c)} = -\frac{9}{8}\pi h'(0)\Omega_3 dx' - \frac{\pi}{4}\Omega_3 \text{Tr}[c(dx_n)\Psi]dx'. \quad (2.52)$$

Then the sum of cases (b) and (c) is zero and Φ is zero. Then we get

Theorem 2.10 *Let M be a 4-dimensional compact spin manifold with the boundary ∂M and the metric g^M as above, then*

$$\widetilde{\text{Wres}}[(\pi^+ D_\Psi^{-1})^2] = 4\pi^2 \int_M \text{Tr} \left[-\frac{1}{12}s - \frac{1}{2}\Psi c(e_i)\Psi c(e_i) + \Psi^2 \right] d\text{vol}_M. \quad (2.53)$$

In [Wa2], we proved the Kastler-Kalau-Walze theorem associated to Dirac operators for 6-dimensional spin manifolds with boundary. In fact, our computations are correct for general Laplacians, then we get

Theorem 2.11 *Let M be a 6-dimensional compact Riemannian manifold with the boundary ∂M and the metric g^M as above and Δ be a general Laplacian acting sections of the vector bundle V , then*

$$\widetilde{\text{Wres}}[(\pi^+ \Delta^{-1})^2] = 8\pi^3 \int_M \text{Tr} \left[\frac{s}{6} + E \right] d\text{vol}_M. \quad (2.54)$$

Since D_Ψ^2 is a general Laplacian, then we get

Corollary 2.12 *Let M be a 6-dimensional compact spin manifold with the boundary ∂M and the metric g^M as above, then*

$$\widetilde{\text{Wres}}[(\pi^+ D_\Psi^{-2})^2] = 8\pi^3 \int_M \text{Tr} \left[-\frac{1}{12}s - \frac{1}{2}\Psi c(e_i)\Psi c(e_i) + 2\Psi^2 \right] d\text{vol}_M. \quad (2.55)$$

In the above two cases, the boundary terms vanish. In the following, we will give a boundary term nonvanishing case and compute $\text{Wres}((D_\Psi D)^{-1})$.

$$\begin{aligned} D_\Psi D &= -g^{ij}\partial_i\partial_j + (-2\sigma^j + \Gamma^j + \Psi c(\partial^j))\partial_j \\ &+ g^{ij}[-\partial_i(\sigma_j) - \sigma_i\sigma_j + \Gamma_{ij}^k\sigma_k + \Psi c(\partial_i)\sigma_j] + \frac{1}{4}s. \end{aligned} \quad (2.56)$$

And

$$\omega_i = \sigma_i - \frac{1}{2}\Psi c(\partial_i), \quad (2.57)$$

$$\begin{aligned} E &= -\Psi c(\partial_i)\sigma^i - \frac{1}{4}s + \frac{1}{2}\partial^j[\Psi c(\partial_j)] \\ &- \frac{1}{2}\Gamma^k\Psi c(\partial_k) + g^{ij} \left[\frac{1}{2}\sigma_i\Psi c(\partial_j) + \frac{1}{2}\Psi c(\partial_i)\sigma_j - \frac{1}{4}\Psi c(\partial_i)\Psi c(\partial_j) \right]. \end{aligned} \quad (2.58)$$

$$E = -\frac{1}{4}s + \frac{1}{2}\nabla_{e_i}^S(\Psi)c(e_i) - \frac{1}{4}\Psi c(e_i)\Psi c(e_i). \quad (2.59)$$

$$D_\Psi D = -[g^{ij}(\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\nabla_{\partial_i}^L\partial_j})] + \frac{1}{4}s - \frac{1}{2}\nabla_{e_i}^S(\Psi)c(e_i) + \frac{1}{4}\Psi c(e_i)\Psi c(e_i). \quad (2.60)$$

Then we get

Theorem 2.13 *Let M be a 4-dimensional compact spin manifold without the boundary, then*

$$\text{Wres}[(D_\Psi D)^{-1}] = 4\pi^2 \int_M \text{Tr} \left[-\frac{1}{12}s + \frac{1}{2}\nabla_{e_i}^S(\Psi)c(e_i) - \frac{1}{4}\Psi c(e_i)\Psi c(e_i) \right] d\text{vol}_M. \quad (2.61)$$

When Ψ is a one form, we can get the following corollary:

Corollary 2.14 *Let M be a 4-dimensional compact spin manifold without the boundary and Ψ be a one form, then*

$$\text{Wres}[(D_\Psi D)^{-1}] = 16\pi^2 \int_M \left[-\frac{1}{12}s + \frac{1}{2}\delta(\Psi) - 2|\Psi|^2 \right] d\text{vol}_M. \quad (2.62)$$

Now we compute $\widetilde{\text{Wres}}[\pi^+ D_\Psi^{-1} \pi^+ D^{-1}]$. We have cases (a) and (b) are the same with Theorem 2.10, but case (c) $= -\frac{9}{8}\pi h'(0)\Omega_3 dx'$, then we get

$$\int_{\partial M} \Phi = \frac{\pi}{4}\Omega_3 \int_{\partial M} \text{Tr}[c(dx_n)\Psi] d\text{vol}_{\partial M}, \quad (2.63)$$

and

Theorem 2.15 *Let M be a 4-dimensional compact spin manifold with the boundary, then*

$$\begin{aligned} \widetilde{\text{Wres}}[\pi^+ D_\Psi^{-1} \pi^+ D^{-1}] &= 4\pi^2 \int_M \text{Tr} \left[-\frac{1}{12}s + \frac{1}{2}e_i(\Psi)c(e_i) - \frac{1}{4}\Psi c(e_i)\Psi c(e_i) \right] d\text{vol}_M \\ &\quad + \frac{\pi}{4}\Omega_3 \int_{\partial M} \text{Tr}[c(dx_n)\Psi] d\text{vol}_{\partial M}. \end{aligned} \quad (2.64)$$

Remark. When Ψ is not a one-form, then boundary term vanishes. When $\Psi = K dx_n$ near the boundary where K is the extrinsic curvature, then the boundary term is proportional to the gravitational action on the boundary. In fact, the reason of the boundary term being not zero is that $\pi^+ D_\Psi$ and $\pi^+ D$ are not symmetric.

3 The Kastler-Kalau-Walze type theorem for conformal perturbations of Dirac operators

In [CM], Connes and Moscovici defined a twisted spectral triple and considered the conformal Dirac operator $e^h D e^h$. In this section, we will prove the Kastler-Kalau-Walze theorem for conformal Dirac operators. That is, we will compute $\text{Wres}[f D^{-1} g D^{-1}]$ for nonzero smooth functions f, g .

$$\begin{aligned} \text{Wres}[f D^{-1} g D^{-1}] &= \text{Wres}[(f^{-1} D g^{-1} D)^{-1}] \\ &= \text{Wres} \left\{ (f^{-1} g^{-1} D^2 + f^{-1} [D, g^{-1}] D)^{-1} \right\} \\ &= \int_M f g \text{wres}[(D^2 - g^{-1} c(dg) D)^{-1}], \end{aligned} \quad (3.1)$$

where wres denotes the residue density and we note that the Kastler-Kalau-Walze theorem is correct pointwisely.

$$\begin{aligned} D^2 - g^{-1} c(dg) D &= -g^{ij} \partial_i \partial_j + [-2\sigma^j + \Gamma^j - g^{-1} c(dg) c(\partial^j)] \partial_j \\ &\quad + [-\partial^j \sigma_j - \sigma^j \sigma_j + \Gamma^k \sigma_k + \frac{1}{4} s - g^{-1} c(dg) c(\partial^j) \sigma_j]. \end{aligned} \quad (3.2)$$

$$\omega_i = \sigma_i + \frac{1}{2} g^{-1} c(dg) c(\partial_i), \quad (3.3)$$

$$\begin{aligned} E &= -\frac{s}{4} + g^{-1} c(dg) c(\partial^j) \sigma_j - \partial^j [\frac{1}{2} g^{-1} c(dg) c(\partial_j)] - \frac{1}{2} \sigma^j g^{-1} c(dg) c(\partial_j) \\ &\quad - \frac{1}{2} g^{-1} c(dg) c(\partial_i) \sigma^i - \frac{1}{4} g^{ij} g^{-1} c(dg) c(\partial_i) g^{-1} c(dg) c(\partial_j) + \frac{1}{2} g^{-1} c(dg) c(\partial_k) \Gamma^k. \end{aligned} \quad (3.4)$$

Since E is globally defined, we compute it in the normal coordinates.

$$\text{Tr}(E)(x_0) = \text{Tr} \left[-\frac{s}{4} - \frac{1}{2} \partial_j (g^{-1} c(dg)) c(\partial_j) - \frac{1}{4} g^{-1} c(dg) c(\partial_i) g^{-1} c(dg) c(\partial_i) \right] (x_0). \quad (3.5)$$

$$\begin{aligned} &\text{Tr} [g^{-1} c(dg) c(\partial_i) g^{-1} c(dg) c(\partial_i)] (x_0) \\ &= g^{-2} \text{Tr} \left[\sum_{i,k,l} \frac{\partial g}{\partial x_k} \frac{\partial g}{\partial x_l} c(\partial_k) c(\partial_i) c(\partial_l) c(\partial_i) \right] \\ &= g^{-2} \text{Tr} \left[\sum_{i,k} \left(\frac{\partial g}{\partial x_k} \right)^2 c(\partial_k) c(\partial_i) c(\partial_k) c(\partial_i) \right] \\ &= g^{-2} \text{Tr} \left[\sum_{i \neq k} \left(\frac{\partial g}{\partial x_k} \right)^2 c(\partial_k) c(\partial_i) c(\partial_k) c(\partial_i) + \sum_k \left(\frac{\partial g}{\partial x_k} \right)^2 c(\partial_k)^4 \right] \end{aligned}$$

$$= g^{-2}(2-n) \sum_k \left(\frac{\partial g}{\partial x_k} \right)^2 \text{Tr}[\text{Id}] \quad (3.6)$$

Similarly,

$$\text{Tr} [\partial_j (g^{-1} c(dg)) c(\partial_j)] = \sum_j \left[\frac{1}{g^2} \left(\frac{\partial g}{\partial x_j} \right)^2 - g^{-1} \frac{\partial^2 g}{\partial x_j^2} \right] \quad (3.7)$$

So

$$\text{Tr} \left[\frac{s}{6} + E \right] = -\frac{s}{3} + 2g^{-1} \sum_j \frac{\partial^2 g}{\partial x_j^2} = -\frac{s}{3} - 2g^{-1} \Delta(g). \quad (3.8)$$

And we get

Theorem 3.1 *Let M be a 4-dimensional compact spin manifold without boundary, then*

$$\text{Wres}[fD^{-1}gD^{-1}] = -4\pi^2 \int_M f \left[\frac{gs}{3} + 2\Delta(g) \right] \text{dvol}_M. \quad (3.9)$$

When f, g have zero points, (3.9) is also correct. Now we consider manifolds with boundary and we will compute $\widetilde{\text{Wres}}[\pi^+(fD^{-1})\pi^+(gD^{-1})]$. As in [Wa1], we have five cases.

$$\begin{aligned} \text{case (a) I} &= -fg \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ q_{-1} \times \partial_{x'}^\alpha \partial_{\xi_n} q_{-1}](x_0) d\xi_n \sigma(\xi') dx' \\ &- f \sum_{j < n} \partial_j(g) \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi_j} \pi_{\xi_n}^+ q_{-1} \times \partial_{\xi_n} q_{-1}](x_0) d\xi_n \sigma(\xi') dx', \end{aligned} \quad (3.10)$$

By the case (a) I) in [Wa1], then

$$-fg \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ q_{-1} \times \partial_{x'}^\alpha \partial_{\xi_n} q_{-1}](x_0) d\xi_n \sigma(\xi') dx' = 0 \quad (3.11)$$

$$\begin{aligned} \pi_{\xi_n}^+ q_{-1} &= \sqrt{-1} \pi_{\xi_n}^+ \left(\frac{c(\xi') + \xi_n c(dx_n)}{1 + \xi_n^2} \right) \\ &= \frac{1}{2\pi} \int_{\Gamma^+} \frac{\frac{c(\xi') + \eta_n c(dx_n)}{(\eta_n + i)(\xi_n - \eta_n)}}{\eta_n - i} d\eta_n = \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)}. \end{aligned} \quad (3.12)$$

$$\partial_{\xi_n} q_{-1} = \sqrt{-1} \left[\frac{c(dx_n)}{|\xi|^2} - \frac{2\xi_n c(\xi)}{|\xi|^4} \right]. \quad (3.13).$$

By $\text{Tr}[c(dx_j)c(dx_n)] = 0$ for $j < n$ and $\int_{|\xi'|=1} \xi_j \sigma(\xi') = 0$ for $j < n$, then

$$-f \sum_{j < n} \partial_j(g) \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi_j} \pi_{\xi_n}^+ q_{-1} \times \partial_{\xi_n} q_{-1}](x_0) d\xi_n \sigma(\xi') dx' = 0, \quad (3.14)$$

and the case (a) I) is zero.

$$\begin{aligned} \text{case (a) II} &= -\frac{1}{2}fg \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ q_{-1} \times \partial_{\xi_n}^2 q_{-1}](x_0) d\xi_n \sigma(\xi') dx' \\ &\quad - \frac{1}{2}g \partial_{x_n} f \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ q_{-1} \times \partial_{\xi_n}^2 q_{-1}](x_0) d\xi_n \sigma(\xi') dx', \end{aligned} \quad (3.15)$$

By the case (a) II) in [Wa1], we get

$$-\frac{1}{2}fg \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ q_{-1} \times \partial_{\xi_n}^2 q_{-1}](x_0) d\xi_n \sigma(\xi') dx' = -\frac{3}{8}\pi h'(0)\Omega_3 f g dx'. \quad (3.16)$$

$$\partial_{\xi_n}^2 q_{-1} = \sqrt{-1} \left(-\frac{6\xi_n c(dx_n) + 2c(\xi')}{|\xi|^4} + \frac{8\xi_n^2 c(\xi)}{|\xi|^6} \right). \quad (3.17)$$

By Lemma 2.2.4 in [Wa1], then

$$\text{trace}[\pi_{\xi_n}^+ q_{-1} \times \partial_{\xi_n}^2 q_{-1}](x_0)|_{\xi'=1} = \frac{-4i}{(\xi_n - i)(\xi_n + i)^3}, \quad (3.18)$$

and

$$\int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ q_{-1} \times \partial_{\xi_n}^2 q_{-1}](x_0) d\xi_n \sigma(\xi') = \pi i \Omega_3. \quad (3.19)$$

So

$$\text{case (a) II} = -\frac{3}{8}\pi h'(0)\Omega_3 f g dx' - \frac{\pi i}{2}\Omega_3 g \partial_{x_n}(f) dx'. \quad (3.20)$$

$$\begin{aligned} \text{case (a) III} &= -\frac{1}{2}fg \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ q_{-1} \times \partial_{\xi_n} \partial_{x_n} q_{-1}](x_0) d\xi_n \sigma(\xi') dx' \\ &\quad - \frac{1}{2}f \partial_{x_n} g \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{\xi_n}^+ q_{-1} \times \partial_{\xi_n} q_{-1}](x_0) d\xi_n \sigma(\xi') dx', \end{aligned} \quad (3.21)$$

Similar to case (a) II),

$$\text{case (a) III} = \frac{3}{8}\pi h'(0)\Omega_3 f g dx' + \frac{\pi i}{2}\Omega_3 f \partial_{x_n}(g) dx'. \quad (3.22)$$

As in [Wa1], the sum of cases (b) and (c) is zero. Then we obtain

$$\int_{\partial M} \Phi = \frac{\pi i \Omega_3}{2} \int_{\partial M} [f \partial_{x_n}(g) - g \partial_{x_n}(f)]_{x_n=0} d\text{vol}_{\partial M}. \quad (3.23)$$

Theorem 3.2 *Let M be a 4-dimensional compact spin manifold with boundary, then*

$$\widetilde{\text{Wres}}[\pi^+(fD^{-1})\pi^+(gD^{-1})] = -4\pi^2 \int_M f \left[\frac{gs}{3} + 2\Delta(g) \right] d\text{vol}_M$$

$$+\frac{\pi i \Omega_3}{2} \int_{\partial M} [f \partial_{x_n}(g) - g \partial_{x_n}(f)]|_{x_n=0} \text{dvol}_{\partial M}. \quad (3.24)$$

When $f = 1$, $g = x_n K$ near the boundary, we have the boundary term is proportional to the gravitational action on the boundary.

4 The spectral action for perturbations of Dirac operators

In [IL], Iochum and Levy studied the spectral action for Dirac operators with one form perturbations and proved that there were no tadpoles for compact spin manifolds without boundary. In [SZ], they investigated the spectral action for scalar perturbations of Dirac operators. In [HPS], Hanisch, Pfäffle and Stephan derived a formula for the gravitational part of the spectral action for Dirac operators on 4-dimensional spin manifolds with totally anti-symmetric torsion. In fact Dirac operators with totally anti-symmetric torsion are three form perturbations of Dirac operators. In this section, We will give some details on the spectral action for Dirac operators with scalar perturbations and we also compute the spectral action for Dirac operators with two form perturbations on 4-dimensional spin manifolds.

For the perturbed self-adjoint Dirac operator D_Ψ , we will calculate the bosonic part of the spectral action. It is defined to be the number of eigenvalues of D_Ψ in the interval $[-\Lambda, \Lambda]$ with $\Lambda \in \mathbf{R}^+$. As in [CC1], it is expressed as

$$I = \text{tr} F \left(\frac{D_\Psi^2}{\Lambda^2} \right). \quad (4.1)$$

Here tr denotes the operator trace in the L^2 completion of $\Gamma(S(F))$, and $F : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is a cut-off function with support in the interval $[0, 1]$ which is constant near the origin. Let $\dim M = n$. By Theorem 2.1, we have the heat trace asymptotics for $t \rightarrow 0$,

$$\text{tr}(e^{-tD_\Psi^2}) \sim \sum_{n \geq 0} t^{n-\frac{m}{2}} a_{2n}(D_\Psi^2). \quad (4.2)$$

One uses the Seeley-deWitt coefficients $a_{2n}(D_\Psi^2)$ and $t = \Lambda^{-2}$ to obtain an asymptotics for the spectral action when $\dim M = 4$ [CC1]

$$I = \text{tr} F \left(\frac{D_\Psi^2}{\Lambda^2} \right) \sim \Lambda^4 F_4 a_0(D_\Psi^2) + \Lambda^2 F_2 a_2(D_\Psi^2) + \Lambda^0 F_0 a_4(D_\Psi^2) \quad \text{as } \Lambda \rightarrow \infty \quad (4.3)$$

with the first three moments of the cut-off function which are given by $F_4 = \int_0^\infty s F(s) ds$, $F_2 = \int_0^\infty F(s) ds$ and $F_0 = F(0)$. Let

$$\Omega_{ij} = \nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i} - \nabla_{[e_i, e_j]}. \quad (4.4)$$

We use [Gi, Thm 4.1.6] to obtain the first three coefficients of the heat trace asymptotics:

$$a_0(D_\Psi) = (4\pi)^{-\frac{n}{2}} \int_M \text{tr}(\text{Id}) d\text{vol}, \quad (4.5)$$

$$a_2(D_\Psi) = (4\pi)^{-\frac{n}{2}} \int_M \text{tr}[\frac{s}{6} + E] d\text{vol}, \quad (4.6)$$

$$a_4(D_\Psi) = \frac{(4\pi)^{-\frac{n}{2}}}{360} \int_M \text{tr}[-12R_{ijij,kk} + 5R_{ijij}R_{klkl} - 2R_{ijik}R_{ljl k} + 2R_{ijkl}R_{ijkl} - 60R_{ijij}E + 180E^2 + 60E_{,kk} + 30\Omega_{ij}\Omega_{ij}] d\text{vol}. \quad (4.7)$$

When $\Psi = f$, by (2.17) and (4.6),

$$a_2(D_f) = (2\pi)^{-\frac{n}{2}} [-\frac{s}{12} + (n-1)f^2]. \quad (4.8)$$

$$5s^2 + 60sE + 180E^2 = \frac{5}{4}s^2 - 30(n-1)sf^2 + 180(n-1)^2f^4. \quad (4.9)$$

$\text{Tr}[\Omega_{ij}\Omega_{ij}]$ is globally defined, thus we only compute it in normal coordinates about x_0 and the local orthonormal frame e_i obtained by parallel transport along geodesics from x_0 . Then

$$\omega_{st}(x_0) = 0, \quad \partial_i(c(e_j)) = 0, \quad [e_i, e_j](x_0) = 0. \quad (4.10)$$

We know that the curvature of the canonical spin connection is $R^S = -\frac{1}{4}R_{ijst}^M c(e_s)c(e_t)$.

$$\begin{aligned} \Omega(e_i, e_j)(x_0) &= [e_i + \frac{1}{4} \sum_{s,t} \omega_{st}(e_i) c(e_s) c(e_t) - f c(e_i)] [e_j + \frac{1}{4} \sum_{s,t} \omega_{st}(e_j) c(e_s) c(e_t) - f c(e_j)] \\ &\quad - [e_j + \frac{1}{4} \sum_{s,t} \omega_{st}(e_j) c(e_s) c(e_t) - f c(e_j)] [e_i + \frac{1}{4} \sum_{s,t} \omega_{st}(e_i) c(e_s) c(e_t) - f c(e_i)] \\ &= -\frac{1}{4} R_{ijst}^M c(e_s) c(e_t) - e_i(f) c(e_j) + e_j(f) c(e_i) + 2f^2 c(e_i) c(e_j), \text{ for } i \neq j. \end{aligned} \quad (4.11)$$

So

$$\begin{aligned} \text{Tr}[\Omega_{ij}\Omega_{ij}](x_0) &= \sum_{i \neq j} \text{Tr} \left\{ \frac{1}{16} R_{ijst}^M R_{ijst_1t_1}^M c(e_s) c(e_t) c(e_{s_1}) c(e_{t_1}) + e_i(f)^2 c(e_j)^2 \right. \\ &\quad \left. + e_j(f)^2 c(e_i)^2 + 4f^4 c(e_i) c(e_j) c(e_i) c(e_j) \right. \\ &\quad \left. - \frac{f^2}{2} R_{ijst}^M [c(e_s) c(e_t) c(e_i) c(e_j) + c(e_i) c(e_j) c(e_s) c(e_t)] \right. \\ &\quad \left. - e_i(f) e_j(f) [c(e_j) c(e_i) + c(e_i) c(e_j)] \right\}. \end{aligned} \quad (4.12)$$

By (2.23), we obtain

$$\text{Tr}[\frac{1}{16} R_{ijst}^M R_{ijst_1t_1}^M c(e_s) c(e_t) c(e_{s_1}) c(e_{t_1})] = -\frac{d}{8} (R_{ijst}^M)^2, \quad (4.13)$$

$$\sum_{i \neq j} \text{Tr}[e_i(f)^2 c(e_j)^2 + e_j(f)^2 c(e_i)^2] = 2d(1-n) \sum_i e_i(f)^2 = 2d(1-n)|df|^2, \quad (4.14)$$

$$\sum_{i \neq j} \text{Tr}[4f^4 c(e_i) c(e_j) c(e_i) c(e_j)] = -4dn(n-1)f^4, \quad (4.15)$$

$$\sum_{i \neq j} \text{Tr}[-e_i(f) e_j(f) (c(e_j) c(e_i) + c(e_i) c(e_j))] = 0, \quad (4.16)$$

$$\sum_{i \neq j} \text{Tr} \left\{ -\frac{f^2}{2} R_{ijst}^M [c(e_s) c(e_t) c(e_i) c(e_j) + c(e_i) c(e_j) c(e_s) c(e_t)] \right\} = -2f^2 ds. \quad (4.17)$$

By (4.12-4.17), we obtain

$$\text{Tr}[\Omega_{ij} \Omega_{ij}] = -\frac{d}{8} (R_{ijst}^M)^2 + 2d(1-n)|df|^2 - 2f^2 ds - 4dn(n-1)f^4. \quad (4.18)$$

By (4.7) (4.9) and (4.18), we get

Theorem 4.1 ([SZ])

$$a_4(D_f) = \frac{d}{360 \times (4\pi)^{\frac{n}{2}}} \left[3\Delta s + \frac{5}{4}s^2 - 30(n+1)sf^2 + 60(n-1)(n-3)f^4 \right. \\ \left. - 2R_{ijik} R_{ljlk} - \frac{7}{4} R_{ijst}^2 + 60(1-n)|df|^2 - 60(n-1)\Delta(f^2) \right]. \quad (4.19)$$

Now we let Ψ is a two form and $\Psi = \sum_{k,l} a_{kl} e^k \wedge e^l$ where $a_{kl} = -a_{lk}$. We may consider $\sqrt{-1}\Psi$ for selfadjoint perturbed Dirac operators. By Corollary 2.4, we obtain

$$a_2(D_\Psi) = d(4\pi)^{-\frac{n}{2}} \left[-\frac{s}{12} + (6-2n)|\Psi|^2 \right]. \quad (4.20)$$

Firstly, we compute $\text{Tr}(E^2)$. By (2.22) and (4.10),

$$e_j(a_{kl}) [c(e_k) c(e_l) c(e_j) - c(e_j) c(e_k) c(e_l)] = 4e_k(a_{kl}) c(e_l), \quad (4.21)$$

$$\begin{aligned} & a_{kl} [c(e_i) c(e_k) c(e_l) + c(e_k) c(e_l) c(e_i)] a_{k_1 l_1} [c(e_i) c(e_{k_1}) c(e_{l_1}) + c(e_{k_1}) c(e_{l_1}) c(e_i)] \\ &= \sum_{i \neq k, l} 2a_{kl} c(e_k) c(e_l) c(e_i) \sum_{i \neq k_1, l_1} 2a_{k_1 l_1} c(e_i) c(e_{k_1}) c(e_{l_1}) \\ &= -4 \sum_{i \neq k, l, k_1, l_1} a_{kl} a_{k_1 l_1} c(e_k) c(e_l) c(e_{k_1}) c(e_{l_1}) \\ &= - \left(\sum_{k \neq l \neq k_1 \neq l_1} + \sum_{k=k_1, l \neq l_1} + \sum_{k=l_1, l \neq k_1} + \sum_{l=k_1, k \neq l_1} + \sum_{l=l_1, k \neq k_1} \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=k_1, l=l_1} + \sum_{k=l_1, l=k_1} \Big) [a_{kl}a_{k_1l_1}c(e_k)c(e_l)c(e_{k_1})c(e_{l_1})] \\
& = -4(n-4) \sum_{k \neq l \neq k_1 \neq l_1} a_{kl}a_{k_1l_1}c(e_k)c(e_l)c(e_{k_1})c(e_{l_1}) \\
& \quad -16(n-3) \sum_{l \neq l_1} a_{kl}a_{kl_1}c(e_l)c(e_{l_1}) + 8(n-2)a_{kl}^2.
\end{aligned} \tag{4.22}$$

Similar to (4.22), we have

$$[a_{kl}c(e_k)c(e_l)]^2 = \sum_{k \neq l \neq k_1 \neq l_1} a_{kl}a_{k_1l_1}c(e_k)c(e_l)c(e_{k_1})c(e_{l_1}) + 4 \sum_{l \neq l_1} a_{kl}a_{kl_1}c(e_l)c(e_{l_1}) - 2a_{kl}^2. \tag{4.23}$$

Then

$$\begin{aligned}
E & = -\frac{s}{4} + 2e_k(a_{kl})c(e_l) + (n-5) \sum_{k \neq l \neq k_1 \neq l_1} a_{kl}a_{k_1l_1}c(e_k)c(e_l)c(e_{k_1})c(e_{l_1}) \\
& \quad + (4n-16) \sum_{l \neq l_1} a_{kl}a_{kl_1}c(e_l)c(e_{l_1}) + (6-2n)a_{kl}^2.
\end{aligned} \tag{4.24}$$

Now we can compute $\text{Tr}(E^2)$.

$$\text{Tr}[2e_k(a_{kl})c(e_l)2e_{k_1}(a_{k_1l_1})c(e_{l_1})] = -4de_k(a_{kl})e_{k_1}(a_{k_1l_1}) = -d|\delta\Psi|^2(x_0), \tag{4.25}$$

$$\text{Tr}\left[\left(-\frac{s}{4} + (6-2n)|\Psi|^2\right)^2\right] = d\left[\frac{s^2}{16} + (n-3)s|X|^2 + (2n-6)^2|X|^4\right], \tag{4.26}$$

$$\begin{aligned}
& \text{tr} \left[\sum_{l \neq l_1} a_{kl}a_{kl_1}c(e_l)c(e_{l_1}) \sum_{p \neq q} a_{k_1p}a_{k_1q}c(e_p)c(e_q) \right] \\
& = a_{kl}a_{kl_1}a_{k_1p}a_{k_1q}(-\delta_l^p\delta_{l_1}^q + \delta_l^q\delta_{l_1}^p)\text{Tr}[\text{Id}] = 0,
\end{aligned} \tag{4.27}$$

$$\sum_{k \neq l \neq k_1 \neq l_1} \sum_{p \neq q} \text{Tr}[a_{kl}a_{k_1l_1}c(e_k)c(e_l)c(e_{k_1})c(e_{l_1})a_{rp}a_{rq}c(e_p)c(e_q)] = 0, \tag{4.28}$$

$$\begin{aligned}
& \sum_{k \neq l \neq k_1 \neq l_1} \sum_{p \neq q \neq r \neq t} \text{Tr}[a_{kl}a_{k_1l_1}c(e_k)c(e_l)c(e_{k_1})c(e_{l_1})a_{pq}a_{rt}c(e_p)c(e_q)c(e_r)c(e_t)] \\
& = \sum_{\{k, l, k_1, l_1\} = \{p, q, r, t\}} \text{Tr}[a_{kl}a_{k_1l_1}c(e_k)c(e_l)c(e_{k_1})c(e_{l_1})a_{pq}a_{rt}c(e_p)c(e_q)c(e_r)c(e_t)] \\
& = 8d|\Psi|^4 - 16da_{kl}a_{k_1l_1}a_{kk_1}a_{ll_1} = 8d|\Psi|^4 - di_{e_k}i_{e_l}(\Psi)i_{e_{k_1}}i_{e_{l_1}}(\Psi)i_{e_k}i_{e_{k_1}}(\Psi)i_{e_l}i_{e_{l_1}}(\Psi).
\end{aligned} \tag{4.29}$$

By (4.24-4.29), we get

$$\text{Tr}(E^2) = d \left[-|\delta\Psi|^2 + \frac{s^2}{16} + (n-3)s|\Psi|^2 + 4(3n^2 - 26n + 59)|\Psi|^4 \right]$$

$$-(n-5)^2 i_{e_k} i_{e_l} (\Psi) i_{e_{k_1}} i_{e_{l_1}} (\Psi) i_{e_k} i_{e_{k_1}} (\Psi) i_{e_l} i_{e_{l_1}} (\Psi) \Big]. \quad (4.30)$$

In the following, we compute $\text{Tr}[\Omega(e_i, e_j) \Omega(e_i, e_j)](x_0)$ under normal coordinates and $n = 4$.

$$\begin{aligned} \nabla_{e_i} &= e_i + \frac{1}{4} \sum_{s,t} \omega_{st}(e_i) c(e_s) c(e_t) - \sum_{k,l \neq i} a_{kl} c(e_k) c(e_l) c(e_i) \\ \Omega(e_i, e_j)(x_0) &= -\frac{1}{4} R_{ijst}^M c(e_s) c(e_t) - e_i \left[\sum_{k,l \neq j} a_{kl} c(e_k) c(e_l) c(e_j) \right] \\ &\quad + e_j \left[\sum_{k,l \neq i} a_{kl} c(e_k) c(e_l) c(e_i) \right] \\ &\quad + \sum_{k,l \neq i} a_{kl} c(e_k) c(e_l) c(e_i) \sum_{k_1, l_1 \neq j} a_{k_1 l_1} c(e_{k_1}) c(e_{l_1}) c(e_j) \\ &\quad - \sum_{k_1, l_1 \neq j} a_{k_1 l_1} c(e_{k_1}) c(e_{l_1}) c(e_j) \sum_{k,l \neq i} a_{kl} c(e_k) c(e_l) c(e_i). \end{aligned} \quad (4.31)$$

In the last two terms in (4.31), we note that $i \neq j$, $k, l \neq i$, $k_1, l_1 \neq j$, $k \neq l$, $k_1 \neq l_1$ and $n = 4$. We divide into three cases.

Case I) $k, l \neq j$ Then $\{k, l, i, j\} = \{1, 2, 3, 4\}$ and $\{k_1, l_1\} \subset \{k, l, i\}$. There are six possibilities as follows 1) $k = k_1$, $l = l_1$ 2) $k_1 = l$, $l_1 = k$ 3) $k_1 = k$, $l_1 = i$ 4) $k_1 = i$, $l_1 = k$ 5) $k_1 = l$, $l_1 = i$ 6) $k_1 = i$, $l_1 = l$

$$\text{Case I)} = \sum_{k,l \neq i, j, i \neq j} [4c(e_j) c(e_i) a_{kl}^2 + 8a_{kl} a_{ki} c(e_l) c(e_j)]. \quad (4.32)$$

Case II) $k = j$, thus $l \neq j$. Then $l, i, k_1, l_1 \neq j$ and at least a pair equals. In $\{l, i, k_1, l_1\}$, if the index i emerges twice, then $k_1 = i$ or $l_1 = i$. If the index i emerges once, then $l = k_1$ or $l = l_1$. We can get

$$\text{Case II)} = 4 \sum_{l \neq l_1 \neq i \neq j} a_{jl} a_{il_1} c(e_l) c(e_{l_1}) + 4 \sum_{l \neq k_1 \neq i \neq j} a_{jl} a_{k_1 l} c(e_i) c(e_{k_1}). \quad (4.33)$$

Case III) $l = j$, thus $k \neq j$. Similar to case II), we obtain

$$\text{Case III)} = 4 \sum_{k \neq l_1 \neq i \neq j} a_{jk} a_{il_1} c(e_k) c(e_{l_1}) + 4 \sum_{k \neq l_1 \neq i \neq j} a_{jk} a_{l_1 k} c(e_i) c(e_{l_1}). \quad (4.34)$$

So

$$\begin{aligned} C &:= \sum_{k,l \neq i} a_{kl} c(e_k) c(e_l) c(e_i) \sum_{k_1, l_1 \neq j} a_{k_1 l_1} c(e_{k_1}) c(e_{l_1}) c(e_j) \\ &\quad - \sum_{k_1, l_1 \neq j} a_{k_1 l_1} c(e_{k_1}) c(e_{l_1}) c(e_j) \sum_{k,l \neq i} a_{kl} c(e_k) c(e_l) c(e_i) \end{aligned}$$

$$\begin{aligned}
&= 4 \sum_{k,l \neq i,j,i \neq j} a_{kl}^2 c(e_j) c(e_i) + 8 \sum_{k,l \neq i,j,i \neq j} a_{kl} a_{ki} c(e_l) c(e_j) \\
&+ 8 \sum_{l \neq l_1 \neq i \neq j} a_{jl} a_{il_1} c(e_l) c(e_{l_1}) + 8 \sum_{l \neq k_1 \neq i \neq j} a_{jl} a_{k_1 l} c(e_i) c(e_{k_1}). \quad (4.35)
\end{aligned}$$

Now we compute $\text{Tr}[\Omega(e_i, e_j) \Omega(e_i, e_j)](x_0)$.

$$\text{Tr} \left[\left(-\frac{1}{4} R_{ijst}^M c(e_s) c(e_t) \right)^2 \right] = -\frac{d}{8} R_{ijst}^2, \quad (4.36)$$

$$\text{Tr} \left[\sum_{k,l \neq j} e_i(a_{kl}) c(e_k) c(e_l) c(e_j) \sum_{k_1, l_1 \neq j} e_i(a_{k_1 l_1}) c(e_{k_1}) c(e_{l_1}) c(e_j) \right] = 2d \sum_{i \neq j} \sum_{k,l \neq j} e_i(a_{kl})^2, \quad (4.37)$$

$$16 \text{Tr} \left[\sum_{k \neq l \neq i,j} a_{kl}^2 c(e_j) c(e_i) \sum_{k_1 \neq l_1 \neq i,j} a_{k_1 l_1}^2 c(e_j) c(e_i) \right] = -16d \sum_{i \neq j} \left(\sum_{k,l \neq i,j} a_{kl}^2 \right)^2, \quad (4.38)$$

$$\begin{aligned}
&64 \text{Tr} \left[\sum_{k,l \neq i,j} a_{kl} a_{ki} c(e_l) c(e_j) \sum_{k_1, l_1 \neq i,j} a_{k_1 l_1} a_{k_1 i} c(e_{l_1}) c(e_j) \right] \\
&= -64d \sum_{i \neq j} \sum_{k,l,k_1, \neq i,j} a_{kl} a_{k_1 l} a_{ki} a_{k_1 i}, \quad (4.39)
\end{aligned}$$

$$\begin{aligned}
&64 \text{Tr} \left[\sum_{k \neq l \neq i \neq j} a_{jk} a_{il} c(e_k) c(e_l) \sum_{k_1 \neq l_1 \neq i \neq j} a_{jk_1} a_{il_1} c(e_{k_1}) c(e_{l_1}) \right] \\
&= 64d \sum_{i \neq j} \sum_{k \neq l \neq i \neq j} (-a_{jk}^2 a_{il}^2 + a_{jk} a_{il} a_{jl} a_{ik}). \quad (4.40)
\end{aligned}$$

$$\text{Tr} \left[\sum_{i \neq j} \sum_{k,l \neq j} e_i(a_{kl}) c(e_k) c(e_l) c(e_j) \sum_{k_1, l_1 \neq i} e_j(a_{k_1 l_1}) c(e_{k_1}) c(e_{l_1}) c(e_i) \right] = 0, \quad (4.41)$$

$$2 \text{Tr} \left[-\frac{1}{4} R_{ijst}^M c(e_s) c(e_t) \times C \right] = 4d \sum_{k,l \neq i,j} (R_{ijji} a_{kl}^2 + 4R_{ijlj} a_{kl} a_{ki} + 2R_{ijkl} a_{jk} a_{il}), \quad (4.42)$$

and the trace of the product of the different terms in C is zero, so we get

$$\begin{aligned}
\frac{1}{d} \text{Tr}[\Omega_{ij} \Omega_{ij}](x_0) &= -\frac{1}{8} R_{ijst}^2 + 4 \sum_{i \neq j} \sum_{k,l \neq j} e_i(a_{kl})^2 - 16 \sum_{i \neq j} \left(\sum_{k,l \neq i,j} a_{kl}^2 \right)^2 \\
&- 128 \sum_{i \neq j} \sum_{k,l,k_1, \neq i,j} a_{kl} a_{k_1 l} a_{ki} a_{k_1 i} + 64 \sum_{i \neq j} \sum_{k \neq l \neq i \neq j} (-a_{jk}^2 a_{il}^2 + a_{jk} a_{il} a_{jl} a_{ik})
\end{aligned}$$

$$+4 \sum_{k,l \neq i,j} (R_{ijji}a_{kl}^2 + 4R_{ijlj}a_{kl}a_{ki} + 2R_{ijkl}a_{jk}a_{il}). \quad (4.43)$$

By (4.7) (4.30) and (4.43), we obtain

Theorem 4.2 *Let $\Psi = \sum_{k,l} a_{kl}e^k \wedge e^l$ where $a_{kl} = -a_{lk}$ and $\dim M = 4$, then in the normal coordinates about x_0 ,*

$$\begin{aligned} a_4(D_\Psi)(x_0) = & \frac{1}{1440\pi^2} \left\{ \Delta(3s + 120|\Psi|^2) + \frac{5}{4}s^2 - 2R_{ijlk}R_{ljlk} - \frac{7}{4}R_{ijkl}^2 + 60s|\Psi|^2 \right. \\ & - 180|\delta\Psi|^2 + 2160|\Psi|^4 - 180i_{e_k}i_{e_l}(\Psi)i_{e_{k_1}}i_{e_{l_1}}(\Psi)i_{e_k}i_{e_{k_1}}(\Psi)i_{e_l}i_{e_{l_1}}(\Psi) \\ & + 120 \sum_{i \neq j} \sum_{k,l \neq j} e_i(a_{kl})^2 - 480 \sum_{i \neq j} \left(\sum_{k,l \neq i,j} a_{kl}^2 \right)^2 \\ & - 3840 \sum_{i \neq j} \sum_{k,l,k_1 \neq i,j} a_{kl}a_{k_1l}a_{ki}a_{k_1i} + 1920 \sum_{i \neq j} \sum_{k \neq l \neq i \neq j} (-a_{jk}^2a_{il}^2 + a_{jk}a_{il}a_{jl}a_{ik}) \\ & \left. + 120 \sum_{k,l \neq i,j} (R_{ijji}a_{kl}^2 + 4R_{ijlj}a_{kl}a_{ki} + 2R_{ijkl}a_{jk}a_{il}) \right\}. \quad (4.44) \end{aligned}$$

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